Indian Statistical Institute, Bangalore Centre.
Back-Paper Exam : Differential Equations

Instructor : Yogeshwaran D.

Date : May 27th, 2015.

Answer for 50 points. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments.
If you are citing results from the class/books, mention it clearly.
Convergence of series needs to be justified.

1. Determine whether the following differential equations have a solution and if so is the solution unique.
   (a) $y^{'} = \sqrt{y}$; $y(0) = 0$. (**5**)
   (b) $y^{'} = x^2 sin(y) - yln(x)$; $y(1) = 2$. (**5**)
2. The Chebyshev differential equation is
$$(1 - x^2)y^{''} - xy^{'} + \alpha^2 y = 0.$$
   Determine two independent power-series solutions for the Chebyshev equation and also mention the radius of convergence for each series. Further, show that if $\alpha = n, n \geq 0$, then there is a polynomial solution of the differential equation. (**10**)
3. Consider the nth order ($n \geq 2$) homogeneous equation on an interval $I = [a, b]$:
$$L[y] = \sum_{i=0}^{n} p_i(x)y^{(n-i)} = 0,$$
   where $y^{(k)}$ denotes the kth derivative of y and $p_0(x) \equiv 1$. Assume that the functions p_i's are continuous on I. Let the functions $y_1, \ldots, y_n$

satisfy $L[y_i] = 0$, $\forall i \in \{1, \ldots, n\}$. Define the Wronskian as

$$W(y_1, \ldots, y_n) := |A_{n,n}|,$$

where $A_{n,n}$ is the matrix defined as below :

$$A_{n,n} := \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

(a) Show that either W is identically 0 in I or $\min_{x \in I} |W(x)| > 0$ for $n = 3$. (**4**)

(b) Assuming the above statement for $n \geq 2$, show that if $y_1, \ldots, y_n$ are linearly independent solutions then any function y_g satisfying $L[y_g] = 0$ can be expressed as a linear combination of $y_1, \ldots, y_n$. (**6**)

4. In the IVPs $\mathbf{x}' = A\mathbf{x}(t)$ with the coefficient matrix A given below, explain the behaviour of the solution as $t \to \infty$. (**10**)

(a)

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}, \ \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} -1 & 1 \\ -5 & 4 \end{pmatrix}, \ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

5. Consider the heat equation

$$a^2 u_{xx} = u_t \ u(0,t) = T_1, \ u(L,t) = T_2, \ x \in (0, L), \ t > 0,$$

$$u(x, 0) = f(x), \ u_t(x, 0) = g(x).$$

Assume that $v(x)$ is the the steady-state solution to the above equation (i.e., $v_t(x,t) = 0$) and the actual solution is $u(x,t) = v(x) + w(x,t)$. Show that $w(x,t)$ satisfies the heat equation with zero temperature boundary conditions. Further, find $v(x)$ and by method of seperation of variables, find $w(x,t)$. Lastly, show that $u(x,t) \to v(x)$ as $t \to \infty$. (**10**)

6. Let D be the open ball of unit radius in $\mathbb{R}^2$ and ∂D denote its boundary i.e., the unit circle. Let $f \in C(\partial D)$ and $U \in C^2(D)$. Firstly show that the following Dirichlet problem

$$\Delta U(x,y) = 0, \ (x,y) \in D, \ U(x,y) = f(x,y) for (x,y) \in \partial D$$

in polar co-ordinates is

$$V_{rr}(r,\theta) + \frac{1}{r^2} V_{\theta\theta}(r,\theta) + \frac{1}{r} V_r(r,\theta) = 0, \ V(1,\theta) = \phi(\theta) = f(cos\theta, sin\theta).$$

Now, solve the above PDE using seperation of variables. (**10**)